CRITICAL DIMENSIONALITY FOR NORMAL FLUCTUATIONS OF MACROVARIABLES IN NONEQUILIBRIUM STATES

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By examining the spatial dimensionality dependence of the scaling behaviour of macrovariables and their fluctuations, the condition for normal fluctuations in non-equilibrium systems is examined. It is found that there is a critical dimension, above which the fluctuations are normal and exhibit Gaussian-Markov behaviour, and below which the fluctuations are non-linear non-Gaussian.

Scaling Method:

We take the macrovariables to be the slowly varying local densities $A_\mu(\mathbf{r}, t)$ appropriate to the system in question, with a cutoff b much longer than microscopic lengths. These obey

$$\dot{A}_\mu(\mathbf{r}, t) = -h_{\mu\mathbf{r}}(A) + R_\mu(\mathbf{r}, t) \tag{1}$$

The $R_\mu(\mathbf{r}, t)$ are fluctuating forces generaled by the elimination of rapidly varying degrees of freedom, and obey

$$< R_\mu(\mathbf{r}, t)\,;a>_\circ = 0 \quad ; \quad < R_\mu(\mathbf{r}, t)\, R_\nu(\mathbf{r}', t')\,; a>_\circ = 2E_{\mu\nu}(\mathbf{r}, \mathbf{r}'\,; a)\,\delta(t - t') \tag{2}$$

where $<\ ; a>_\circ$ means the conditional average over a stationary ensemble with the values of the A fixed to be a.

The $A_\mu(\mathbf{r}, t)$ are split into $y_\mu(\mathbf{r}, t)$, obeying the appropriate deterministic equations of motion, plus the fluctuations $Z_\mu(\mathbf{r}, t)$ which obey

$$\dot{Z}_\mu(\mathbf{r}, t) = -\triangle h_{\mu\mathbf{r}}(Z\,; y) + R_\mu(\mathbf{r}, t)$$

where $$-\triangle h_{\mu\mathbf{r}}(Z\,; y) = h_{\mu\mathbf{r}}(y + Z) - h_{\mu\mathbf{r}}(y) \tag{3}$$

In the scaling method all lengths $\geqslant$ b are scaled by a factor L, while microscopic lengths ℓ_m (lengths $<$ b) are left invariant. The behaviour of y_μ and Z_μ is then examined in the limit $L \to \infty$ (i.e.: $b/\ell_m \to \infty$, or the " large scale limit "). Scaling exponents α_μ and β_μ for y_μ and Z_μ, along with time scaling exponents τ and θ are introduced by writing:

$$\begin{aligned} &\text{In the } y_\mu \text{ equation:} \quad && y_\mu \to L^{-\alpha_\mu} y_\mu \quad ; \quad t \to L^{\tau} t \\ & && \qquad\qquad\qquad\qquad (L \gg 1) \\ &\text{In the } Z_\mu \text{ equation:} \quad && Z_\mu \to L^{-\beta_\mu} Z_\mu \quad ; \quad t \to L^{\theta} t \end{aligned} \tag{4}$$

We define further scale exponents:

$$\begin{aligned} &\Delta h_{\mu r} \to L^{-\beta_\mu - \theta_{h_\mu}} \Delta h_{\mu r} \quad ; \quad E_{\mu\nu} \to L^{-d-(\psi_\mu + \psi_\nu)/2} E_{\mu\nu} \\ &g(\mathbf{r}, \mathbf{r}') \to L^{\nu_\mu - d} g(\mathbf{r}, \mathbf{r}') \qquad \text{(correlation function)} \end{aligned} \tag{5}$$

α_μ and τ are determined by requiring that the macroscopic equations be invariant under scaling. Fluctuation dissipation theorems, and the assumption that the probability distribution is invariant yields.

$$\theta = \mathrm{Min}_\mu [\theta_{h\mu}] \quad ; \quad \beta_\mu = (d + \psi_\mu - \theta)/2 \quad ; \quad \beta_\mu = (d - \gamma_\mu)/2 \tag{6}$$

CRITICAL DIMENSIONALITY:

$$\begin{aligned} &\text{For large L, } Z_\mu / y_\mu \to g_\mu(L) \cdot Z_\mu(L{=}1) / y_\mu(L{=}1) \quad \text{where} \\ &g_\mu(L) = L^{(d-\Delta_\mu)/2} \quad , \quad \Delta_\mu = 2\alpha_\mu + \gamma_\mu \end{aligned} \tag{7}$$

Define critical dimensionality d_c = lower bound of the region $\{d: d > \Delta = \max_\mu [\Delta_\mu]\}$. If $d > d_c$, $g_\mu(L) \to 0$ as $L \to \infty$, and the fluctuations become negligible compared to the deterministic part, and normal Gaussian-Markov behaviour is obeyed. If $d \leqslant d_c$ this is not true.

EXAMPLE 1 SCHLÖGL MODEL CHEMICAL REACTION

$$B + X \underset{k_3}{\overset{k_1}{\rightleftharpoons}} C \quad , \quad A + X \underset{k_0}{\overset{k_2}{\rightleftharpoons}} 2X \quad \text{with [A], [B], [C] constant and uniform.}$$

The macrovariable is the concentration $x = [x(\mathbf{r}, t)]$, with cutoff b much longer than the mean distance between reactive collisions. The deterministic part y_μ obeys

$$\dot{y}(\mathbf{r},t) = D\nabla^2 y(\mathbf{r},t) + B + a\,y(\mathbf{r},t) - c\,y(\mathbf{r},t)^2 \tag{8}$$

and the fluctuation obeys

$$\dot{Z}(\mathbf{r},t) = D\ {}^2 Z(\mathbf{r},t) - \gamma Z(\mathbf{r},t) - c\,Z(\mathbf{r},t)^2 + R(\mathbf{r},t) \tag{9}$$

where $\quad B = k_3\,[C]\,,\ a = k_2\,[A] - k_1\,[B]\,,\ c = k_4\,,\ \gamma = (a^2 + 4cB)^{\nu^2}$

$$< R(\mathbf{r},t)\,R(\mathbf{r}',t') > = \frac{1}{2}\,[\,B + (k_1 + k_2)\,x(\mathbf{r}) + cx(\mathbf{r})^2 + 2D\nabla r_L \nabla r' x(\mathbf{r})\,]\ \ (\mathbf{r}-\mathbf{r}')\,\nu\,(t-t') \tag{10}$$

Equation (9) defines a characteristic length $\xi = \sqrt{D/\nu}$ which diverges at the critical point $a = B = q = 0$.

In the non-critical region $\xi \ll b$, so ξ is not scaled, which leads to

$$\alpha = \tau = 0 \quad ; \quad \theta = \psi = \gamma = 0,\ \beta = d/2$$
$$d_c = \Delta = 0 \tag{11}$$

Thus in the non-critical region fluctuations are normal in all dimensions, and the equation for $Z(\mathbf{r},t)$ may be linearized to represent normal Gaussian Markov behaviour.

In the critical region $\xi \gg b$, so ξ is scaled, which gives (assuming $\beta \geqslant \alpha$)

$$\alpha = \tau = 2\ ;\ \theta = \psi = 2,\ \gamma = 0,\ \beta = d/2$$
$$d_c = \Delta = 4 \tag{12}$$

Thus only for $d > 4$ are the fluctuations normal. Below $d = 4$ the non-linear term in (9) is important.

EXAMPLE 2 FLUCTUATIONS IN LAMINAR HYDRODYNAMIC FLOW

The deterministic part of the local velocity $\mathbf{u}(\mathbf{r}, t)$ obeys (for an incompressible fluid):

$$\left(\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\mathbf{u} \tag{13}$$

The fluctuation obeys:

$$\left(\frac{\partial}{\partial t} + \mathbf{u}.\nabla\right)\mathbf{Z} + \mathbf{Z}.\nabla\mathbf{u} + \mathbf{Z}.\nabla\mathbf{Z} = -\frac{1}{\rho_0}\nabla(\Delta p) + \nu\nabla^2\mathbf{Z} + \mathbf{R}(\mathbf{r}, t) \tag{14}$$

ρ_0 is density, p is pressure, ν is viscosity, $\Delta p = p(y+z) - p(y)$, and $\langle RR' \rangle$ is given by the usual (e.g. Landau and Lifchitz) expression.

Applying scaling and requiring a dissipative balance (i.e.: the viscous terms balance the inertial terms) yields (far from any critical regions):–

$$\begin{gathered}\alpha_\mu = 1,\ \tau = 2;\ \theta = \psi_\mu = 2,\ \gamma_\mu = 0,\ \beta_\mu = d/2 \\ d_c = \Delta = 2\end{gathered} \tag{15}$$

SCALING NEAR THE ONSET OF A SPATIAL PATTERN (e.g. THE BENARD PROBLEM)

Near this hydrodynamic instability, a certain mode exhibits a very large correlation length ξ in the horizontal direction only. If k_c is the critical wavevector, a cutoff Q_c is imposed on $|k - k_c|$, with $B = 1/Q_c \gg \ell_M$, the vertical width of the fluid layer. The scaling is then applied in the horizontal direction only:–

$$r_{HORIZ}\ (\geqslant B) \rightarrow L\, r_{HORIZ},\ \xi \rightarrow L\xi,\ \ \ell_M \rightarrow \ell m \tag{16}$$

Then for $d > d_c$.,

$$\begin{gathered}\alpha_\mu = 1,\ \tau = 2;\ \theta = \psi_\mu = 2,\ \beta_\mu = (d-2)/2 \\ d_c = \Delta = 4\end{gathered} \tag{17}$$

Note:

In the above examples there are two kinds of critical fluctuations, even though d_c is the same in each case.

(a) In the Schlögl reaction, β_μ is the same (d/2) in both the non-critical region and the critical region when $d > d_c$, whereas α_μ changes from 0 to 2.

(b) In the Benard problem, α_μ is the same (1) in both regions, while β_μ changes from d/2 to (d − 2)/2.

不安定系の異常揺動と緩和

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一変数の場合の不安定点近傍における緩和とゆらぎを一般的にとり扱うスケーリング理論[1)~4)]の要点を最初に復習し，それを多モードに拡張する。[5)~6)] その場合にもやはり，異常揺動定理が導かれる。多成分レーザー模型及び化学反応系への応用例を述べる。[6)] 超放射への応用[7)]については，有光・鈴木の報告を参照して下さい。

基本的な考え方は，一変数でも多モード系で同じであるから

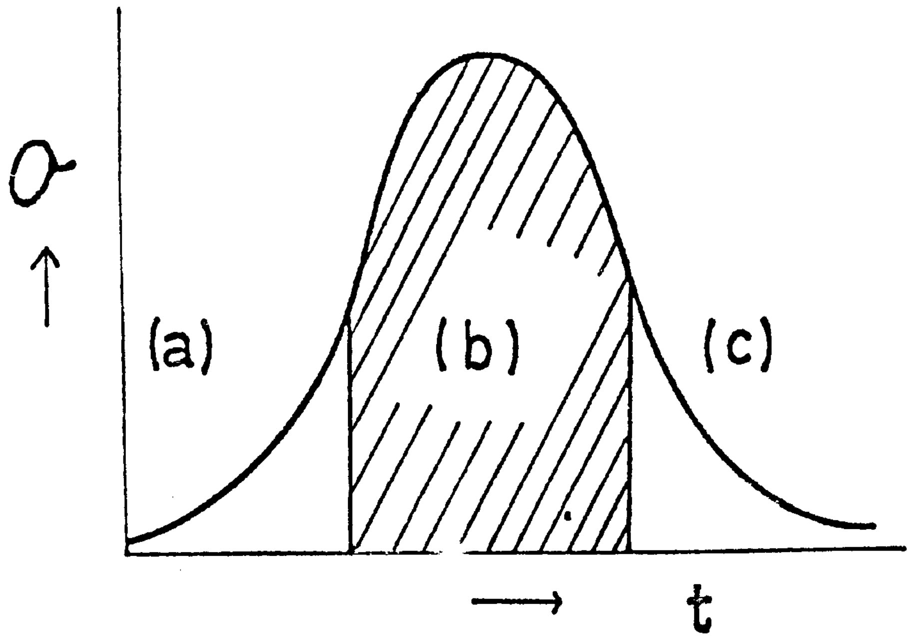

図 1.　σ: ゆらぎ，(a) 初期領域，(b) 第 2 非線型領域，(c) 終領域